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2007 J. Phys. A: Math. Theor. 40 8385

(http://iopscience.iop.org/1751-8121/40/29/013)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 8385–8391 doi:10.1088/1751-8113/40/29/013

On the invariant hyperplanes for *d*-dimensional polynomial vector fields

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Received 14 December 2006, in final form 4 June 2007
Published 3 July 2007
Online at stacks.iop.org/JPhysA/40/8385

Abstract

We deal with polynomial vector fields $\mathcal{X}$ of the form $\sum_{k=1}^{d} P_k(x_1, \ldots, x_d)\partial/\partial x_k$ with $d \geqslant 2$. Let m_k be the degree of P_k. We call $(m_1, \ldots, m_d)$ the degree of $\mathcal{X}$. We provide the best upper bounds for the polynomial vector field $\mathcal{X}$ in the function of its degree $(m_1, \ldots, m_d)$ of (1) the maximal number of invariant hyperplanes, (2) the maximal number of parallel invariant hyperplanes, and (3) the maximal number of invariant hyperplanes that pass through a single point. Moreover, if $m_i = m$, $i = 1, \ldots, d$, we show that these best upper bounds are reached taking into account the multiplicity of the invariant hyperplanes.

PACS numbers: 02.40.Sf, 02.30.Hq, 02.60.Nm
Mathematics Subject Classification: 58F14, 58F22, 34C05

1. Introduction and statement of the results

As usual we denote by $\mathbb{C}[x_1, \ldots, x_d]$ the ring of the polynomials in the variables $x_1, \ldots, x_d$ with coefficients in $\mathbb{C}$. By definition a *polynomial differential system* in $\mathbb{C}^d$ is a system of the form

$$\frac{dx_i}{dt} = P_i(x_1, \ldots, x_d), \qquad i = 1, \ldots, d, \tag{1}$$

where $P_i \in \mathbb{C}[x_1, \ldots, x_d]$. If m_i is the degree of P_i, then we say that $\mathbf{m} = (m_1, \ldots, m_d)$ is the *degree* of the polynomial system. Without loss of generality in the rest of the paper we assume that $m_1 \geqslant \cdots \geqslant m_d$.

We denote by

$$\mathcal{X} = \sum_{i=1}^{d} P_i(x_1, \ldots, x_d)\frac{\partial}{\partial x_i} \tag{2}$$

the polynomial vector field associated with system (1) of degree $\mathbf{m}$.

An *invariant algebraic variety* for system (1) or for the vector field (2) is an algebraic variety $f(x_1, \ldots, x_d) = 0$ with $f \in \mathbb{C}[x_1, \ldots, x_d]$ such that for some polynomial $K \in \mathbb{C}[x_1, \ldots, x_d]$ we have $\mathcal{X} f = \nabla f \cdot \mathcal{X} = Kf$. The polynomial K is called the *cofactor* of the invariant algebraic variety $f = 0$. We remark that if the polynomial system has degree $\mathbf{m} = (m_1, \ldots, m_d)$, with $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_d$, then any cofactor has at most degree $m_1 - 1$.

From $\mathcal{X} f = Kf$ it follows that if a solution curve of system (1) has a point on the algebraic variety $f = 0$, then the whole solution curve is contained in $f = 0$. This is the reason of calling $f = 0$ invariant, because it is invariant by the flow of the system. The converse is true, i.e. assume that we have an algebraic variety $f = 0$ such that if a solution curve of system (1) has a point on it, then the whole solution curve is contained in $f = 0$. Without loss of generality we can assume that the polynomial f is irreducible in $\mathbb{C}[x_1, \ldots, x_d]$. Then since $\mathcal{X} f$ is zero always that $f = 0$ by Hilbert's Nullstellensatz it follows that the polynomial f divides the polynomial $\mathcal{X} f$, the quotient is the cofactor polynomial K.

If the degree of f is 1 then we say that the algebraic variety $f = 0$ is an *invariant hyperplane*.

The knowledge of the invariant algebraic varieties of a differential system provides important information for understanding the dynamics of the system. Thus, if the number of invariant algebraic varieties is sufficiently large then there exists a first integral of the system that can be computed explicitly, see for instance [5]. On the other hand, the invariant algebraic varieties allow us to control better the interesting regions from the dynamical point of view of the system; as an example see [4] where they are used to provide a bounded region where the Lorenz attractor lives. Of course, the simplest use of invariant algebraic varieties is for separating the initial phase space of the system into invariant pieces.

Now we shall introduce one of the best tools in order to look for invariant algebraic varieties. Let $\mathcal{X}$ be a polynomial vector field on $\mathbb{C}^d$ and let W be a finitely generated vector subspace of $\mathbb{C}[x_1, \ldots, x_d]$. The *extactic algebraic variety of $\mathcal{X}$ associated with W* is

$$\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \cdots & \mathcal{X}(v_l) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{X}^{l-1}(v_1) & \mathcal{X}^{l-1}(v_2) & \cdots & \mathcal{X}^{l-1}(v_l) \end{pmatrix} = 0, \tag{3}$$

where $\{v_1, \ldots, v_l\}$ is a basis of W, $l = \dim(W)$ is the dimension of W and $\mathcal{X}^j(v_i) = \mathcal{X}^{j-1}(\mathcal{X}(v_i))$. It is known due to the properties of the determinant and of the derivation that the definition of extactic algebraic variety is independent of the chosen basis of W.

In fact we learn this definition from the paper [8], but this notion goes back to the work of Lagutinskii at the beginning of the twentieth century; see the references quoted in [3]. We have used the definition of $\mathcal{E}_W(\mathcal{X})$ in different papers (see [2, 6]).

The notion of extactic algebraic variety $\mathcal{E}_W(\mathcal{X})$ is important in this paper for two reasons. First it allows us to detect when an algebraic variety $f = 0$ with $f \in W$ is invariant by the polynomial vector field $\mathcal{X}$; see the next result proved in [2, 6] for polynomial vector fields in $\mathbb{C}^2$ and $\mathbb{R}^3$, respectively, but its extension to $\mathbb{C}^d$ is clear. In any case, since its proof is short and easy, for completeness we will present it at the beginning of section 2. The second reason why $\mathcal{E}_W(\mathcal{X})$ is important in this paper is because it allows us to define and compute easily the multiplicity of an invariant algebraic variety, and in particular of an invariant hyperplane.

Proposition 1. *Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{C}^d$ and let W be a finitely generated vector subspace of $\mathbb{C}[x_1, \ldots, x_d]$ with $\dim(W) > 1$. Then every algebraic invariant variety $f = 0$ for the vector field $\mathcal{X}$, with $f \in W$, is a factor of $\mathcal{E}_W(\mathcal{X})$.*

By proposition 1, $f = 0$ is an invariant hyperplane of the polynomial vector field $\mathcal{X}$ if the polynomial f is a factor of $\mathcal{E}_W(\mathcal{X})$ with W generated by $\{1, x_1, \ldots, x_d\}$. From [2] *the invariant hyperplane* $f = 0$ *has multiplicity* k if k is the greatest positive integer such that f^k divides the polynomial $\mathcal{E}_W(\mathcal{X})$ with W generated by $\{1, x_1, \ldots, x_d\}$. Again from [2] if the invariant hyperplane $f = 0$ of $\mathcal{X}$ has multiplicity k, then there is a family of vector fields $\mathcal{X}_\varepsilon$ with the same degree, then $\mathcal{X}$ such that $\mathcal{X}_0 = \mathcal{X}$ and for all $\varepsilon > 0$ sufficiently small $\mathcal{X}_\varepsilon$ has k different hyperplanes tending to the hyperplane $f = 0$ when $\varepsilon \to 0$.

When we study the maximal number of invariant hyperplanes through a point that a polynomial vector field $\mathcal{X}$ can have, without loss of generality doing a translation of this point we can assume that this point is the origin. Then the *multiplicity* k *of an invariant hyperplane through the origin* is defined as the greatest positive integer k such that f^k divides the polynomial $\mathcal{E}_W(\mathcal{X})$ with W generated by $\{x_1, \ldots, x_d\}$.

Of course, in dimension 2 an invariant hyperplane is an invariant straight line. The number of invariant straight lines for polynomial vector fields in $\mathbb{R}^2$ has been studied for several authors, see for instance [10]. Additionally, it is known that for polynomial vector fields of degree $(2, 2)$ the maximal number of invariant straight lines is 5. Recently Zhang Xiang [11] and Sokulski [9] proved that the maximal number of real invariant straight lines for polynomial vector fields in $\mathbb{R}^2$ of degrees $(3, 3)$ and $(4, 4)$ are 8 and 9, respectively. As far as we know, the main results about this number for polynomial vector fields in $\mathbb{R}^2$ are summarized in the following theorem proved in [1].

Theorem 2. *Assume that a polynomial vector field* $\mathcal{X}$ *in* $\mathbb{R}^2$ *of degree* $\mathbf{m} = (m, m)$ *has finitely many invariant straight lines. Then the following statements hold.*

(a) The number of invariant straight lines of $\mathcal{X}$ *is at most* $3m - 1$.
(b) The number of parallel invariant straight lines of $\mathcal{X}$ *is at most* m.
(c) The number of different invariant straight lines of $\mathcal{X}$ *through a single point is at most* $m + 1$.

The goal of this paper is to improve theorem 2 and extend it to polynomial vector fields in $\mathbb{C}^d$. Thus our main two results are the following ones.

Theorem 3. *Assume that a polynomial vector field* $\mathcal{X}$ *in* $\mathbb{C}^d$ *with* $d \geqslant 2$ *of degree* $\mathbf{m} = (m_1, \ldots, m_d)$ *with* $m_1 \geqslant \cdots \geqslant m_d$ *has finitely many invariant hyperplanes. Then the following statements hold.*

(a) The number of invariant hyperplanes of $\mathcal{X}$ *taking into account its multiplicity is at most*

$$\left(\sum_{k=1}^{d} m_k\right) + \binom{d}{2}(m_1 - 1). \tag{4}$$

(b) The number of parallel invariant hyperplanes of $\mathcal{X}$ *taking into account its multiplicity is at most* m_1.
(c) The number of different invariant hyperplanes of $\mathcal{X}$ *through a single point taking into account its multiplicity is at most*

$$\left(\sum_{k=1}^{d-1} m_k\right) + \binom{d-1}{2}(m_1 - 1) + 1. \tag{5}$$

Theorem 3 is proved in section 2.

Clearly there are polynomial vector fields $\mathcal{X}$ with infinitely many invariant hyperplanes. It is sufficient to take in system (1) the polynomial $P_d(x_1, \ldots, x_d) = 0$.

Theorem 4. *Assume that a polynomial vector field $\mathcal{X}$ in $\mathbb{C}^d$ with $d \geqslant 2$ of degree $\mathbf{m} = (m, \ldots, m)$ has finitely many invariant hyperplanes. Then the following statements hold.*

(a) The number of invariant hyperplanes of $\mathcal{X}$ taking into account its multiplicity is at most

$$dm + \binom{d}{2}(m-1),$$

and this upper bound is reached for some $\mathcal{X}$.

(b) The number of parallel invariant hyperplanes of $\mathcal{X}$ taking into account its multiplicity is at most m, and this upper bound is reached for some $\mathcal{X}$.

(c) The number of different invariant hyperplanes of $\mathcal{X}$ through a single point taking into account its multiplicity is at most

$$(d-1)m + \binom{d-1}{2}(m-1) + 1,$$

and this upper bound is reached for some $\mathcal{X}$.

Theorem 4 is proved in section 3.

Note that showing that the upper bounds of theorem 4 are the best possible, we are also showing that when $m_i = m, i = 1, \ldots, d$, then the bounds in theorem 3 are exact.

Note that theorem 2 is now a particular case of theorem 4 when $d = 2$. Of course, the binomial number $\binom{d-1}{2}$ when $d = 2$ is zero by definition.

We remark that we have examples of real polynomial vector fields of degree (4, 4) with 11 complex invariant straight lines, but that only with real invariant straight lines we cannot reach more than 9 of such lines according to the results of [9, 11]. More information about the number of real invariant straight lines for polynomial vector fields in $\mathbb{R}^2$ can be found in [1].

Some preliminary results about the number of invariant hyperplanes of polynomial vector fields in $\mathbb{R}^d$ can be found in [7], and in $\mathbb{C}^d$ in [12]. But the upper bounds founded in these papers are not the best ones with the exception of the one corresponding to the statement (a) of theorem 3. But also in that case they did not take into account the multiplicity of the invariant hyperplanes and they do not prove that the bound is the best one.

2. Proof of theorem 3

First we shall prove proposition 1 and after theorem 3.

Proof of proposition 1. Suppose that we are in the assumptions of proposition 1. Then let $f = 0$ be an invariant algebraic variety of $\mathcal{X}$ such that $f \in W$. As was observed, the choice of the basis of W plays no role in the definition of extactic curve, therefore we can take $v_1 = f$ in (3). Since

$$\begin{aligned}
&\mathcal{X}(f) = K_f f,\\
&\mathcal{X}^2(f) = \mathcal{X}(K_f f) = \left(\mathcal{X}(K_f) + K_f^2\right) f,\\
&\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&\mathcal{X}^k(f) = \mathcal{X}(\mathcal{X}^{k-1}(f)) = \cdots = (\text{polynomial}) f,
\end{aligned}$$

f is a factor of the polynomial $\mathcal{E}_W(\mathcal{X})$. $\square$

Proof of theorem 3. Suppose that we are in the assumptions of theorem 3. We define W as the $\mathbb{C}$-vector subspace of $\mathbb{C}[x_1, \ldots, x_d]$ generated by $1, x_1, \ldots, x_d$. Then if $f \in W$ we have that $f = 0$ is a hyperplane if f is not a constant.

By proposition 1 if $f = 0$ is a hyperplane of $\mathcal{X}$, then f is a factor of

$$\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_d \\ 0 & \mathcal{X}(x_1) & \mathcal{X}(x_2) & \cdots & \mathcal{X}(x_d) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \mathcal{X}^d(x_1) & \mathcal{X}^d(x_2) & \cdots & \mathcal{X}^d(x_d) \end{pmatrix}. \tag{6}$$

Note that for $k = 1, \ldots, d$, the maximum degrees of the polynomials $\mathcal{X}(x_k)$, $\mathcal{X}^2(x_k)$, $\mathcal{X}^3(x_k), \ldots, \mathcal{X}^d(x_k)$ are m_k, $m_1 - 1 + m_k$, $2(m_1 - 1) + m_k, \ldots, (d-1)(m_1 - 1) + m_k$, respectively. Taking into account that $m_1 \geqslant \cdots \geqslant m_d$ and the definition of the determinant, it follows that the maximum degree of the polynomial $\mathcal{E}_W(\mathcal{X})$ is

$$\begin{aligned}&[(d-1)(m_1 - 1) + m_1] + [(d-2)(m_1 - 1) + m_2] \\ &\qquad + [(d-3)(m_1 - 1) + m_3] + \cdots + [m_1 - 1 + m_{d-1}] + m_d,\end{aligned}$$

i.e. the degree of $\mathcal{E}_W(\mathcal{X})$ is given by expression (4). Note that the previous degree corresponds to the degree of the polynomial $\mathcal{X}^d(x_1)\mathcal{X}^{d-1}(x_2)\cdots\mathcal{X}^2(x_{d-1})\mathcal{X}(x_d)$, which is one of the polynomials of determinant (6) corresponding to a permutation of d elements with a maximal degree.

Since the polynomial $\mathcal{E}_W(\mathcal{X})$ can have at most as many factors of the form $a_0 + a_1x_1 + \cdots + a_dx_d$ as its degree, by proposition 1 it follows statement (a) of theorem 3.

If we have a set of parallel hyperplanes in $\mathbb{C}^d$ doing a convenient linear change of coordinates, we can assume that all the equations of these hyperplanes are of the form $x_1 - \text{constant} = 0$. In other words all these hyperplanes can be written in the form $f = 0$ with $f \in W$, where W is the $\mathbb{C}$-vector subspace of $\mathbb{C}[x_1, \ldots, x_d]$ generated by 1 and x_1. Therefore, by proposition 1 if $f = 0$ is one of these hyperplanes of $\mathcal{X}$, then f is a factor of

$$\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix} 1 & x_1 \\ 0 & \mathcal{X}(x_1) \end{pmatrix}.$$

Since the degree of $\mathcal{X}(x_1) = P_1(x_1, \ldots, x_d)$ is m_1, statement (b) of theorem 3 is proved. □

Doing a translation of the coordinates (if necessary) we can assume that the single point—through it passes a set of hyperplanes—is located at the origin of coordinates. So all these hyperplanes are of the form $a_1x_1 + \cdots + a_dx_d = 0$. That is all these hyperplanes can be written in the form $f = 0$ with $f \in W$, where W is the $\mathbb{C}$-vector subspace of $\mathbb{C}[x_1, \ldots, x_d]$ generated by $x_1, \ldots, x_d$. Therefore, by proposition 1 if $f = 0$ is one of these hyperplanes of $\mathcal{X}$, then f must be a factor of

$$\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix} x_1 & x_2 & \cdots & x_d \\ \mathcal{X}(x_1) & \mathcal{X}(x_2) & \cdots & \mathcal{X}(x_d) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{X}^{d-1}(x_1) & \mathcal{X}^{d-1}(x_2) & \cdots & \mathcal{X}^{d-1}(x_d) \end{pmatrix}. \tag{7}$$

Note that for $k = 1, \ldots, d$ the degree of the polynomials x_k, $\mathcal{X}(x_k)$, $\mathcal{X}^2(x_k)$, $\mathcal{X}^3(x_k), \ldots, \mathcal{X}^{d-1}(x_k)$ are $1, m_k, m_1 - 1 + m_k, 2(m_1 - 1) + m_k, \ldots, (d-2)(m_1 - 1) + m_k$, respectively. Taking into account that $m_1 \geqslant \cdots \geqslant m_d$ and the definition of the determinant, it follows that the degree of the polynomial $\mathcal{E}_W(\mathcal{X})$ is

$$[(d-2)(m_1-1)+m_1]+[(d-3)(m_1-1)+m_2] \\ +[(d-4)(m_1-1)+m_3]+\cdots+[m_1-1+m_{d-2}]+m_{d-1}+1,$$

i.e. the degree of $\mathcal{E}_W(\mathcal{X})$ is given by expression (5). Note that the previous degree corresponds to the degree of the polynomial $\mathcal{X}^{d-1}(x_1)\mathcal{X}^{d-2}(x_2)\cdots\mathcal{X}(x_{d-1})x_d$, which is one of the polynomials of determinant (7) corresponding to a permutation of d elements with a maximal degree.

Again since the polynomial $\mathcal{E}_W(\mathcal{X})$ can have at most as many factors of the form $a_1x_1+\cdots+a_dx_d$ as its degree, by proposition 1 it follows statement (c) of theorem 3.

3. Proof of theorem 4

We assume that we are under the hypotheses of theorem 4. The first parts of statements (a), (b) and (c) of theorem 4 follow directly from theorem 3 putting $m_1=\cdots=m_d=m$. Therefore, we must prove for these three statements that the corresponding upper bounds are reached for convenient polynomial vector fields.

We consider the polynomial differential system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}=x_i^m, \qquad i=1,\ldots,d. \tag{8}$$

For this system using induction with respect to d it follows that polynomial (6) becomes

$$a(m)\left(\prod_{k=1}^{d}x_k^m\right)\left(\prod_{1\leqslant i<j\leqslant d}\left(x_j^{m-1}-x_i^{m-1}\right)\right),$$

where $a(m)$ is a constant which only depends on m.

Clearly system (8) has the hyperplane $x_k=0$ invariant with multiplicity m for $k=1,\ldots,d$. Moreover, it is easy to check that the algebraic variety $x_j^{m-1}-x_i^{m-1}=0$ is invariant by system (8) with cofactor $(m-1)\left(x_i^{m-1}+x_j^{m-1}\right)$ for all i and j such that $1\leqslant i<j\leqslant d$. Note that the homogeneous polynomial $x_j^{m-1}-x_i^{m-1}$ factorizes as product of $m-1$ linear homogenous polynomials in $\mathbb{C}[x_i,x_j]$. Now we need the following well-known result, see for instance [5].

Proposition 5. *We suppose that $f\in\mathbb{C}[x_1,\ldots,x_d]$ and let $f=f_1^{n_1}\cdots f_r^{n_r}$ be its factorization in irreducible factors over $\mathbb{C}[x_1,\ldots,x_d]$. Then, for a polynomial system (1), $f=0$ is an invariant algebraic variety with cofactor K_f if and only if $f_i=0$ is an invariant algebraic variety for each $i=1,\ldots,r$ with cofactor K_{f_i}. Moreover, $K_f=n_1K_{f_1}+\cdots+n_rK_{f_r}$.*

By proposition 5 every linear factor of $x_i^{m-1}-x_j^{m-1}$ in $\mathbb{C}[x_1,\ldots,x_d]$ is an invariant hyperplane with multiplicity 1. In short, the total number of invariant hyperplanes of system (8) taking into account their multiplicities is

$$dm+\binom{d}{2}(m-1).$$

Hence the second part of statement (a) of theorem 4 is proved.

System (1) with $P_1(x_1,\ldots,x_d)=\prod_{k=1}^{m}(x_1-k)$ has exactly m parallel invariant hyperplanes, namely $x_1=k$ for $k=1,\ldots,m$. This completes the proof of the second part of statement (b) of theorem 4. In fact for proving this statement we also can use system (8), because for that system $x_1=0$ is a parallel hyperplane to itself with multiplicity m.

Again we consider system (8). For this system using induction with respect to d it follows that polynomial (7) becomes

$$b(m)\left(\prod_{k=1}^{d} x_k\right)\left(\prod_{1\leqslant i<j\leqslant d}\left(x_j^{m-1}-x_i^{m-1}\right)\right),$$

where $b(m)$ is a constant which only depends on m.

This system has through the origin the following invariant hyperplanes: x_k for $k = 1, \ldots, d$ with multiplicity 1, and the invariant hyperplanes defined by the linear factors of $x_i^{m-1} - x_j^{m-1}$ for all i and j such that $1 \leqslant i < j \leqslant d$ with multiplicity 1. In short, the total number of invariant hyperplanes through the origin is

$$d + \binom{d}{2}(m-1) = (d-1)m + \binom{d-1}{2}(m-1) + 1,$$

i.e. this number is the maximal number of possible invariant hyperplanes through the origin taking into account their multiplicity. Therefore we have shown the second part of statement (c) of theorem 4.

Acknowledgments

The first author is partially supported by a DGICYT grant number MTM2005-06098-C02-01 and by a CICYT grant number 2005SGR 00550. The second author is partially supported by a CNPq grant number 620029/2004-8. All authors are also supported by the joint project CAPES–MECD grant 071/04 and HBP2003-0017, respectively.

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